Regularization of Grad's 13 Moment Equations Using a Hermite Polynomial Representation of Velocity Distribution Function

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Abstract. In the paper we suggest a new closure for Grad's thirteen moment equations for medium rarefied gases by using a Hermite polynomial approximation for the monatomic gas velocity distribution function, and applying the Chapman-Enskog regularization method to Grad's velocity distribution function that corresponds to his thirteen moment equation. In our paper, the collision term of the Boltzmann equation is assumed to be in the Bhatnagar-Gross-Krook (BGK) form. The velocity distribution function for resulting 13 regularized moment equations is presented.

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I. INTRODUCTION

One of the hardest problems in computational fluid dynamics is the modeling of medium rarefied gases with the Knudsen number in the range 0.005 – 1. In this case, the gas is rarefied to such a degree that using the Navier-Stokes-Fourier equations is questionable, but it is not sufficiently rarefied that using Direct Simulation Monte Carlo (DSMC) methods is effective. It should be stressed that in recent years modeling of flows for Knudsen numbers in this range have become more and more important for many practical applications, ranging from the modeling of re-entry of space vehicles into the atmosphere to modeling of microscale flows and heat transfer in microchannels. One of the ways to cover this range of Knudsen numbers is to use thirteen (or more) moment equations instead of the Navier-Stokes-Fourier 5 equations.

In 1949, Grad [1, 2] derived the 13 moment equations corresponding to the second order of the Knudsen number. Unfortunately, Grad's moment equations sometimes produce unphysical solutions; for example, they fail to describe smooth shock structures for Mach numbers above a critical value [3]. In 2003, Struchtrup and Horrihon [4] regularized Grad's 13 moment equations, extending them to third order of the Knudsen number. The authors have developed a new closure method which principally differs from well-known Chapman-Enskog method [5] (that was used to derive a closure of Euler's gas dynamics equations), in which they have not used the Hermite polynomial representation of the velocity distribution function.

In the present paper we suggest a new closure for Grad's 13 moment equations by using a Hermite polynomial approximation for monatomic gas distribution function, and applying the Chapman-Enskog regularization method to Grad's velocity distribution function that corresponds to his 13 moment equations. In our paper, the collision term is assumed to be in BGK form. It must be stressed that the equations obtained by this method differ from the equations obtained in [4]. One of the reasons is that in [4] the authors use 26 non-Hermite polynomials for approximation of the velocity distribution function and correspondingly 26 moments, while in our method we use 29 Hermite polynomials and consequently 29 moments. In other words, since we use the basis for approximation of the velocity distribution function which is non-congruent to the basis in [4], our set of equations must differ from the set of equations obtained in [4]. We explicitly show why our representation of the velocity distribution function has good physical sense. The integral representation for the 13 moments of the Boltzmann equation is presented in Section II, and Hermite polynomial approximation of the velocity distribution function is derived in Section III. Grad's regularized 13 moment equations are obtained in Section IV, and conclusions are presented in Section V.

II. INTEGRAL EQUATIONS FOR HEAT FLUX AND STRESS TENSOR

The phase density of a monatomic ideal gas is described by the Boltzmann equation,

$$\frac{\partial(n\cdot f)}{\partial t}+V_i\cdot\frac{\partial(n\cdot f)}{\partial x_i}=St(n\cdot f)\,,\qquad \int_{\vec V} f\cdot d^3\vec V=1\,, \tag{1}$$

where n is the number density of gas molecules, f is the velocity distribution function, $V_i = (V_x, V_y, V_z)$ is the particle velocity, $x_i = (x, y, z)$ are the particle coordinates, the integral means the integration over the entire velocity space, and $St(n\cdot f)$ is the collision term that accounts for the change in the velocity distribution functions due to collisions. Here we assume elastic collisions.

Let us introduce 13 moments of the particle distribution function: ρ as the mass density of gas molecules, $u_i = (u_x, u_y, u_z)$ as the gas flow gas molecules, V_T as the thermal velocity, $q_i = (q_x, q_y, q_z)$ as the heat flux, and $\sigma_{ij} = (\sigma_{xx}, \sigma_{xy}, \sigma_{xz}, \sigma_{yy}, \sigma_{yz})$ as components of the stress tensors,

$$\rho=m\cdot n\cdot\int_{\vec V} f\cdot d^3\vec V\,,\quad u_i=\int_{\vec V} f\cdot V_i\cdot d^3\vec V\,,\quad \frac{3}{2}\cdot\frac{\rho\cdot V_T^2}{2}=\int_{\vec V} f\cdot(\vec V-\vec u)^2\cdot d^3\vec V\,, \tag{2}$$

$$q_i=\frac{\rho}{2}\cdot\int_{\vec V} f\cdot(V_i-u_i)\cdot(\vec V-\vec u)^2\cdot d^3\vec V\,,\quad \sigma_{ij}=\rho\cdot\int_{\vec V} f\cdot\left((V_i-u_i)\cdot(V_j-u_j)-\delta_{ij}\cdot\frac{V_T^2}{2}\right)\cdot d^3\vec V\,, \tag{3}$$

where m is the mass of a particle. Multiplying Eq. (1) by

$$m\,,\ m\cdot\vec V\,,\ \frac{m}{2}\cdot\vec V^2\,,\ \frac{m}{2}\cdot(\vec V-\vec u)\cdot(\vec V-\vec u)^2\,,\ m\cdot\left((V_i-u_i)\cdot(V_j-u_j)-\delta_{ij}\cdot\frac{V_T^2}{2}\right)\ \text{where}\ (ij)=(xx,xy,xz,yy,yz) \tag{4}$$

and then integrating the obtained equations over the entire velocity space, taking into account that the number of colliding particles, their total momentum, and their total energy are conserved in collisions, after tedious algebra the first five moment equations that correspond to mass, momentum and energy conservation laws can be derived to the form given in [6], and the moment equations for q_i, σ_{xy}, and σ_{xx} are

$$\begin{aligned}&\frac{\partial q_i}{\partial t}+\frac{\partial}{\partial x_k}(u_k\cdot q_i)+q_k\cdot\frac{\partial u_i}{\partial x_k}-\left(\frac{5}{2}\cdot\frac{V_T^2}{2}\right)\cdot\left(\frac{\partial}{\partial x_i}\left(\frac{\rho\cdot V_T^2}{2}+\frac{\partial\sigma_{ij}}{\partial x_j}\right)\right)-\frac{\sigma_{ik}}{\rho}\cdot\left(\frac{\partial}{\partial x_k}\left(\frac{\rho\cdot V_T^2}{2}\right)+\frac{\partial\sigma_{jk}}{\partial x_j}\right)+\\&+\frac{\partial}{\partial x_k}\left(\frac{\rho}{2}\cdot\int_{\vec V}(V_i-u_i)\cdot(\vec V-\vec u)^2\cdot(V_k-u_k)\cdot f\cdot d^3\vec V\right)+\left(\rho\cdot\int_{\vec V}(V_i-u_i)\cdot(V_k-u_k)\cdot(V_j-u_j)\cdot f\cdot d^3\vec V\right)\cdot\frac{\partial u_j}{\partial x_k}=\\&=\frac{\rho}{2}\cdot\int_{\vec V}(V_i-u_i)\cdot(\vec V-\vec u)\cdot St(f)\cdot d^3\vec V,\end{aligned} \tag{5}$$

$$\begin{aligned}&\frac{\partial\sigma_{xy}}{\partial t}+\sigma_{ky}\cdot\frac{\partial u_x}{\partial x_k}+\sigma_{kx}\cdot\frac{\partial u_y}{\partial x_k}+\frac{\partial}{\partial x_k}\left(u_k\cdot\sigma_{xy}\right)+\rho\cdot V_T^2\cdot\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)+\\&+\frac{\partial}{\partial x_k}\left(\rho\cdot\int_{\vec V}(V_k-u_k)\cdot(V_x-u_x)\cdot(V_y-u_y)\cdot f\cdot d^3\vec V\right)=\rho\cdot\int_{\vec V}(V_x-u_x)\cdot(V_y-u_y)\cdot St(f)\cdot d^3\vec V\,,\end{aligned} \tag{6}$$

$$\frac{\partial\sigma_{xx}}{\partial t}-\frac{1}{3}\cdot\rho\cdot V_T^2\cdot\frac{\partial u_i}{\partial x_i}-\frac{2}{3}\cdot\frac{\partial q_i}{\partial x_i}-\frac{2}{3}\cdot\sigma_{ij}\cdot\frac{\partial u_i}{\partial x_i}+\frac{\partial}{\partial x_k}(u_k\cdot\sigma_{xx})+2\cdot\sigma_{xk}\cdot\frac{\partial u_x}{\partial x_k}+\rho\cdot V_T^2\cdot\frac{\partial u_x}{\partial x}+$$
$$+\frac{\partial}{\partial x_k}\left(\rho\cdot\int_{\vec V}\left((V_x-u_x)^2-\frac{V_T^2}{2}\right)\cdot(V_k-u_k)\cdot f\cdot d^3\vec V\right)=\rho\cdot\int_{\vec V}\left((V_x-u_x)^2-\frac{V_T^2}{2}\right)\cdot St(f)\cdot d^3\vec V\,. \tag{7}$$

Changing the order of indexes (*x*, *y*, *z*) to (*x*, *z*, *y*) and (*y*, *z*, *x*) in Eq. (6) we obtain equations for σ_{xz} and σ_{yz} respectively, and to (*y*, *x*, *z*) in Eq. (7) an equation for σ_{yy}.

It is worth noting that there are no collision terms in the moment equations for ρ, u_i, and V_T [7], while equations for heat flux and stress tensor, Eqs. (5) – (7), do include collision terms. Since collision terms can produce new moments, the set of these 13 moment equations in general is not self-contained. However, in the case of Maxwell molecules and BGK approximation of the collision term,

$$St(f)=\frac{f_M-f}{\tau}\quad\text{where}\quad f_M=\left(\pi\cdot V_T^2\right)^{-3/2}\cdot\exp\left(-\frac{(\vec V-\vec u)^2}{V_T^2}\right)\,, \tag{8}$$

the collision terms do not produce any new moments; in Eq. (8) τ is the collision time depending on coordinates and time. In other words, these approximations of the collision term *do not mix the moments*. This is a key point for any theory of moment approximation of the Boltzmann equation. In this paper we will use the BGK collision term; the case of Maxwellian molecules can be described in a similar way.

III. HERMITE POLYNOMIAL APPROXIMATION

We assume here a Hermite polynomial approximation of the velocity distribution function. The velocity distribution function can be described as a combination of three-dimensional Hermite polynomials that correspond to *x*, *y*, *z* directions of the velocity. We represent the velocity distribution via the 29 Hermite polynomials

$$f_H=f_M\cdot\sum_{k=1}^{29}\Lambda_k\cdot\hat H_k(\chi_x,\chi_y,\chi_y)\,,\ \text{where}\quad \chi_i=(V_i-u_i)/V_T\,, \tag{9}$$

$$\hat H_1=H_0\,,\ \hat H_2=\frac{1}{2}\cdot H_{1x}\,,\ \hat H_3=\frac{1}{2}\cdot H_{1y}\,,\ \hat H_4=\frac{1}{2}\cdot H_{1z}\,,\ \hat H_5=\frac{1}{4}\cdot H_{2x}\,,\ \hat H_6=\frac{1}{4}\cdot H_{1x}\cdot H_{1y}\,,\ \hat H_7=\frac{1}{4}\cdot H_{1x}\cdot H_{1z}\,, \tag{10}$$

$$\hat H_8=\frac{1}{4}\cdot H_{2y}\,,\ \hat H_9=\frac{1}{4}\cdot H_{1y}\cdot H_{1z}\,,\ \hat H_{10}=\frac{1}{4}\cdot H_{2z}\,,\ \hat H_{11}=\frac{1}{8}\cdot\left(H_{3x}+H_{1x}\cdot H_{2y}+H_{1x}\cdot H_{2z}\right), \tag{11}$$

$$\hat H_{12}=\frac{1}{8}\cdot\left(H_{3y}+H_{1y}\cdot H_{2z}+H_{1y}\cdot H_{2x}\right),\ \hat H_{13}=\frac{1}{8}\cdot\left(H_{3z}+H_{1z}\cdot H_{2y}+H_{1z}\cdot H_{2x}\right), \tag{12}$$

$$\hat H_{14}=H_{4x}\,,\ \hat H_{15}=H_{4y}\,,\ \hat H_{16}=H_{4z}\,,\ \hat H_{17}=H_{2x}\cdot H_{2y}\,,\ \hat H_{18}=H_{2x}\cdot H_{2z}\,,\ \hat H_{19}=H_{2y}\cdot H_{2z}\,, \tag{13}$$

$$\hat H_{20}=H_{3x}\cdot H_{1y}\,,\ \hat H_{21}=H_{3x}\cdot H_{1z}\,,\ \hat H_{22}=H_{3y}\cdot H_{1x}\,,\ \hat H_{23}=H_{3y}\cdot H_{1z}\,,\ \hat H_{24}=H_{3z}\cdot H_{1x}\,,\ \hat H_{25}=H_{3z}\cdot H_{1y}\,, \tag{14}$$

$$\hat H_{26}=H_{1x}\cdot H_{1y}\cdot H_{2z}\,,\ \hat H_{27}=H_{1y}\cdot H_{1z}\cdot H_{2x}\,,\ \hat H_{28}=H_{1z}\cdot H_{1x}\cdot H_{2z}\,,\ \hat H_{29}=H_{1x}\cdot H_{1y}\cdot H_{1z}\,. \tag{15}$$

It should be stressed that all $\hat H$ polynomials are orthoganals. Since the velocity distribution function f_H has to satisfy Eqs. (2), we obtain that $\Lambda_1=1$, $\Lambda_2=\Lambda_3=\Lambda_4=0$, and $\Lambda_5+\Lambda_8+\Lambda_{10}=0$. Thus, the particle distribution function $n\cdot f_H$ has 29 variables, ρ, u_x, u_y, u_z, V_T, Λ_5 – Λ_9, Λ_{11} – Λ_{29}. It worth noting that the truncated velocity distribution function f_H that consists of the first ten nonzero Hermite polynomials has the form of the Chapman-

Enskog and Grad's velocity distribution functions [1, 2, 6]. In the next section it will be shown why we have selected this representation of velocity distribution function.

IV. A CLOSURE OF GRAD'S 13 MOMENT EQUATIONS

Let us rewrite the equations for heat flux, Eq (5), and stress tensor, Eqs. (5) and (6) for the case of BGK collision term, Eq. (7),

$$\frac{\partial q_i}{\partial t}+\langle\ldots.\rangle=-\frac{q_i}{\tau}\,,\quad \frac{\partial \sigma_{xx}}{\partial t}+\langle\ldots.\rangle=-\frac{\sigma_{xx}}{\tau}\,,\quad \frac{\partial \sigma_{xy}}{\partial t}+\langle\ldots.\rangle=-\frac{\sigma_{xy}}{\tau}\,, \tag{16}$$

where <….> are the left hand side terms in Eqs. (5) – (7) positioned after corresponding time derivatives. Equations for other components of stress tensor, as has been mention in Section II, can be obtained by proper rotatation of indexes. To complete this system of equations, a velocity distribution function has to be chosen. Grad [1, 2] has suggested his velocity distribution function that is the trancated velocity distribution function f_H , Eq. (9), with the first 10 non-zero terms and obtain his set of equations [1, 2].

Let us obtain a set of the 13 regularized Grad moment equations using the Hermite polynomial approximation of the velocity distribution function and the Chapman-Enskog closure method. First let us represent the under integral polynomials in the left-hand sides of Eqs. (5) for q_x and Eqs. (6) and (7) in Hermite form:

$$\chi_x^2\cdot(\chi_x^2+\chi_y^2+\chi_z^2)=\frac{H_{4x}}{16}+\frac{H_{2x}\cdot H_{2y}}{4}+\frac{H_{2x}\cdot H_{2z}}{4}+\frac{7\cdot H_{2x}}{4}+\frac{H_{2y}}{2}+\frac{H_{2z}}{2}+\frac{5\cdot H_0}{2}\,,\ \chi_x^3=\frac{H_{3x}}{8}+\frac{3\cdot H_{1x}}{4}\,, \tag{17}$$

$$\chi_x\cdot\chi_y\cdot(\chi_x^2+\chi_y^2+\chi_z^2)=\frac{H_{3x}\cdot H_{1y}}{16}+\frac{H_{3y}\cdot H_{1x}}{16}+\frac{H_{1x}\cdot H_{1y}\cdot H_{2z}}{16}+\frac{7\cdot H_{1x}\cdot H_{1y}}{8}\,,\ \chi_x^2\cdot\chi_y=\frac{H_{2x}\cdot H_{1y}}{8}+\frac{H_{1y}}{8}\,, \tag{18}$$

$$\chi_x\cdot\chi_z\cdot(\chi_x^2+\chi_y^2+\chi_z^2)=\frac{H_{3x}\cdot H_{1z}}{16}+\frac{H_{3z}\cdot H_{1x}}{16}+\frac{H_{1x}\cdot H_{1z}\cdot H_{2y}}{16}+\frac{7\cdot H_{1x}\cdot H_{1z}}{8}\,,\ \chi_x^2\cdot\chi_z=\frac{H_{2x}\cdot H_{1z}}{8}+\frac{H_{1z}}{8}\,, \tag{19}$$

$$\chi_x\cdot\chi_y\cdot\chi_z=\frac{H_{1x}\cdot H_{1y}\cdot H_{1z}}{8}\,,\ \chi_x\cdot\chi_y^2=\frac{H_{1x}\cdot H_{2y}}{8}+\frac{H_{1x}}{8}\,,\ \left(\chi_x^2-\frac{1}{2}\right)\cdot\chi_y=\frac{H_{2x}\cdot H_{1y}}{8}\,, \tag{20}$$

$$\left(\chi_x^2-\frac{1}{2}\right)\cdot\chi_x=\frac{H_{3x}}{8}+\frac{H_{1x}}{2}\,,\ \left(\chi_x^2-\frac{1}{2}\right)\cdot\chi_z=\frac{H_{2x}\cdot H_{1z}}{8}\,, \tag{21}$$

where variables χ are given in Eq. (9). The Hermite representations of the under integral polynomials in equations for q_y, q_z, σ_{xz}, σ_{yz}, σ_{yy}, can be obtained by proper rotation of indexes in Eqs. (17) – (21). Thus, the complete list of Hermite polynomial that represents under integral polynomials in equations for heat flux and stress tensor is

$$H_{4x},\ H_{4y},\ H_{4z},\ H_{2x}\cdot H_{2y},\ H_{2x}\cdot H_{2z},\ H_{2y}\cdot H_{2z},\ H_{3x}\cdot H_{1y},\ H_{3x}\cdot H_{1z},\ H_{3y}\cdot H_{1x}, \tag{22}$$

$$H_{3y}\cdot H_{1z},\ H_{3z}\cdot H_{1x},\ H_{3z}\cdot H_{1z},\ H_{1x}\cdot H_{1y}\cdot H_{2z},\ H_{1x}\cdot H_{1z}\cdot H_{2x},\ H_{1y}\cdot H_{1z}\cdot H_{2x},\ H_{1x}\cdot H_{1y}\cdot H_{1z}, \tag{23}$$

$$H_{3x},\ H_{3y},\ H_{3z},\ H_{2x}\cdot H_{1y},\ H_{2x}\cdot H_{1z},\ H_{2y}\cdot H_{1x},\ H_{2y}\cdot H_{1z},\ H_{2z}\cdot H_{1x},\ H_{2z}\cdot H_{1y},\ H_{2x},\ H_{2y}, \tag{24}$$

$$H_{2z},\ H_{1x}\cdot H_{1y},\ H_{1x}\cdot H_{1z},\ H_{1y}\cdot H_{1z},\ H_{1x},\ H_{1y},\ H_{1z},\ H_0. \tag{25}$$

Now following the Chapman-Enskog method [5], let us write the velocity distribution function as

$$f = f_{GRAD} + \tau \cdot f_M \cdot f_1, \tag{26}$$

where f_1 has a Hermite form. Since the velocity distribution function $\tau f_M \cdot f_1$ does not have to contribute into the previously obtained 13 moments ρ, u_i, V_T, q_i, and σ_{ij}, it follows that the Hermite polynomials included in the Grad's velocity distribution function, Eq. (10) – (12), have to be excluded from f_1. But as the velocity distribution function $\tau f_M \cdot f_1$ has to contribute into integrals in Eqs. (5) – (7), we obtain that f_1, as follows from Eqs. (22) – (25), has to be a combination only of the Hermite polynomials presented in Eqs. (22) and (23); the Hermite polynomials presented in Eq. (24) and (25) are included in the Grad's velocity distribution function, Eqs. (10) – (12). Subsequently, we obtain that

$$f_1 = \sum_{i=14}^{29} \Lambda_i \cdot \hat{H}_i \,, \tag{27}$$

where functions $\hat{H}$ are shown in Eqs. (13) – (15). Thus, we have shown that the chosen set of Hermite polynomials, Eqs. (9) – (15) has good physical sense for representing the velocity distribution function.

Substituting f_H, Eqs. (27), for f into Eqs. (16) and introducing the following set of M-moments:

$$M_{4i} = \tau \cdot V_T^4 \cdot \rho \cdot \int_{\vec{V}} f_M \cdot f_1 \cdot H_{4i} \cdot d^3\vec{V}, \; i = (x, y, z); \quad M_{2i2j} = V_T^4 \cdot \rho \cdot \int_{2j\vec{V}} f \cdot H_{2i} \cdot H_{2j} \cdot d^3\vec{V}, \; ij = (xy, xz, yz); \tag{28}$$

$$M_{3i1j} = V_T^4 \cdot \rho \cdot \int_{\vec{V}} f \cdot H_{3i} \cdot H_{1j} \cdot d^3\vec{V}, \; ij = (xy, xz, yx, yz, zx, zy); \quad M_{1x1y1z} = V_T^3 \cdot \rho \cdot \int_{2j\vec{V}} f \cdot H_{1x} \cdot H_{1y} \cdot H_{1z} \cdot d^3\vec{V}; \tag{29}$$

$$M_{1i1j2k} = V_T^4 \cdot \rho \cdot \int_{2j\vec{V}} f \cdot H_{1i} \cdot H_{1i} \cdot H_{2k} \cdot d^3\vec{V}, \; ijk = (xyz, yzx, zxy), \tag{30}$$

we obtain the following equations for q_x, σ_{xy}, and σ_{xx}:

$$\frac{\partial q_x}{\partial t} + \{\ldots\} + \frac{\partial}{\partial x}\left(\frac{M_{4x}}{32} + \frac{M_{2x2y}}{8} + \frac{M_{2x2z}}{8}\right) + \frac{\partial}{\partial y}\left(\frac{M_{3x1y}}{32} + \frac{M_{1x3y}}{32} + \frac{M_{1x1y2z}}{32}\right) + \\ + \frac{\partial}{\partial z}\left(\frac{M_{3x1z}}{32} + \frac{M_{1x3z}}{32} + \frac{M_{1x1z2y}}{32}\right) + \frac{M_{1x1y1z}}{8} \cdot \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_y}{\partial z}\right) = -\frac{q_x}{\tau}, \tag{31}$$

$$\frac{\partial \sigma_{xy}}{\partial t} + \{\ldots\} + \frac{1}{8} \cdot \frac{\partial}{\partial z}\left(M_{1x1y1z}\right) = -\frac{\sigma_{xy}}{\tau}, \qquad \frac{\partial \sigma_{xx}}{\partial t} + \{\ldots\} = -\frac{\sigma_{xx}}{\tau}, \tag{32}$$

where {….} are terms due to pure Grad's velocity distribution function [1, 2]. Thus, to obtain equations for the heat flux and the stress tensor we have to derive equations for *M*-moments. As one can see there is no contribution of new *M*-moments in the equation for σ_{xx}. The equations for other components of the heat flux and the stress tensor can be obtained by a proper rotation of the indexes in Eqs. (31) and (32). We have applied the Chapman-Enskog technique and obtained the following expression for M-moments:

$$-\frac{M_{1x1y1z}}{\tau} = 4 \cdot \rho \cdot V_T^2 \cdot \left[\frac{\partial}{\partial x}\left(\frac{\sigma_{yz}}{\rho}\right) + \frac{\partial}{\partial y}\left(\frac{\sigma_{xz}}{\rho}\right) + \frac{\partial}{\partial z}\left(\frac{\sigma_{xy}}{\rho}\right)\right] + \frac{16 \cdot q_z}{15} \cdot \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) + \\ + \frac{16 \cdot q_y}{15} \cdot \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) + \frac{16 \cdot q_x}{15} \cdot \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right), \tag{33}$$

$$-\frac{M_{2x2y}}{\tau}=-\frac{32}{5}\cdot\frac{V_T^2}{\rho}\left(q_x\cdot\frac{\partial\rho}{\partial x}+q_y\cdot\frac{\partial\rho}{\partial y}\right)+\left(\frac{32}{5}\cdot V_T^2\right)\cdot\frac{\partial q_i}{\partial x_i}+\left(\frac{32}{5}\cdot V_T^2\right)\cdot\frac{\partial q_y}{\partial y}+\left(\frac{32}{5}\cdot q_z\right)\cdot\frac{\partial V_T^2}{\partial z}+$$
$$+\left(\frac{64}{5}\cdot q_x\right)\cdot\frac{\partial V_T^2}{\partial x}+\left(\frac{64}{5}\cdot q_y\right)\cdot\frac{\partial V_T^2}{\partial y}+\frac{16}{3}\cdot V_T^2\cdot\left(2\cdot\sigma_{yy}-\sigma_{xx}\right)\cdot\frac{\partial u_x}{\partial x}+ \tag{34}$$
$$+\frac{16}{3}\cdot V_T^2\cdot\left(2\cdot\sigma_{xx}-\sigma_{yy}\right)\cdot\frac{\partial u_y}{\partial y}-\frac{16}{3}\cdot V_T^2\cdot\left(\sigma_{xx}+\sigma_{yy}\right)\cdot\frac{\partial u_z}{\partial z}+\left(32\cdot V_T^2\cdot\sigma_{xy}\right)\cdot\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right),$$

$$-\frac{M_{4x}}{\tau}=-\frac{288\cdot q_x}{5}\cdot\frac{\partial V_T^2}{\partial x}+\frac{96\cdot q_y}{5}\cdot\frac{\partial V_T^2}{\partial y}+\frac{96\cdot q_z}{5}\cdot\frac{\partial V_T^2}{\partial z}+\frac{192\cdot V_T^2}{5}\cdot\frac{\partial q_x}{\partial x}-$$
$$-\frac{192\cdot V_T^2\cdot q_x}{5\cdot\rho}\cdot\frac{\partial\rho}{\partial x}+32\cdot V_T^2\cdot\sigma_{xx}\cdot\left(2\cdot\frac{\partial u_x}{\partial x}-\frac{\partial u_y}{\partial y}-\frac{\partial u_z}{\partial z}\right), \tag{35}$$

$$-\frac{M_{3x1y}}{\tau}=16\cdot V_T^2\cdot\sigma_{xy}\cdot\left(2\cdot\frac{\partial u_x}{\partial x}-\frac{\partial u_y}{\partial y}-\frac{\partial u_z}{\partial z}\right)+24\cdot V_T^2\cdot\sigma_{xx}\cdot\left(\frac{\partial u_y}{\partial x}+\frac{\partial u_x}{\partial y}\right)-$$
$$-\frac{48\cdot V_T^2}{5\cdot\rho}\cdot\left(q_y\cdot\frac{\partial\rho}{\partial x}+q_x\cdot\frac{\partial\rho}{\partial y}\right)+\frac{48\cdot V_T^2}{5}\cdot\left(\frac{\partial q_x}{\partial y}+\frac{\partial q_y}{\partial x}\right)+\frac{48}{5}\cdot\left(q_y\cdot\frac{\partial V_T^2}{\partial x}+q_x\cdot\frac{\partial V_T^2}{\partial y}\right), \tag{36}$$

$$-\frac{M_{1x1y2z}}{\tau}=\frac{16\cdot V_T^2\cdot\sigma_{xy}}{3}\cdot\left(2\cdot\frac{\partial u_z}{\partial z}-\frac{\partial u_x}{\partial x}-\frac{\partial u_y}{\partial y}\right)+8\cdot V_T^2\cdot\sigma_{zz}\cdot\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)+16\cdot V_T^2\cdot\sigma_{yz}\cdot\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right)+$$
$$+16\cdot V_T^2\cdot\sigma_{xz}\cdot\left(\frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y}\right)-\frac{16\cdot V_T^2}{5\cdot\rho}\cdot\left(q_y\cdot\frac{\partial\rho}{\partial x}+q_x\cdot\frac{\partial\rho}{\partial y}\right)+\frac{16\cdot V_T^2}{5}\cdot\left(\frac{\partial q_y}{\partial x}+\frac{\partial q_x}{\partial y}\right)+\frac{16}{5}\cdot\left(q_y\cdot\frac{\partial V_T^2}{\partial x}+q_y\cdot\frac{\partial V_T^2}{\partial y}\right). \tag{37}$$

Substituting M-moment equations into Eqs. (31) and (32) the equation for q_x, and σ_{xy} can be obtained. A proper rotation of indexes in Eqs. (33) – (37) allows one to obtain equations for the rest M-moments.

V. CONCLUSIONS

We have presented a new set of moment equations for rarefied gas dynamics. Our equations are a closure for Grad's 13 moment equations extended to the third order of the Knudsen number. We have assumed a Hermite polynomial approximation for the gas velocity distribution function and the BGK approximation of the collision term in the Boltzmann kinetic equation. We have also used the well-known Chapman-Enskog regularization method [5]. We have shown that the selected 29-term Hermite polynomial representation of the velocity distribution function makes good physical sense. Our equations differ from a similar set of equations obtained by Struchtrup and Torrihon [4]. One of reasons is that in [4], the authors use 26 non-Hermite polynomials for approximation of the velocity distribution function and, correspondingly, 26 moments, while in our method we use 29 Hermite polynomials and, consequently, 29 moments. Analysis of the predictions obtained by using both sets of equations with DSMC simulations, or numerical solutions of BGK model kinetic equations, is needed for verification of the applicability of the both methods. In the future we will attempt to derive a complete set of Grad's 13 regularized moment equations.

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